Vectors and Scalars

- Scalar: A quantity specified by its magnitude only
- Vector: A quantity specified both by its magnitude and direction.

- To distinguish a vector from a scalar quantity, it is usually written with an arrow above it, or in bold to distinguish it from a scalar.

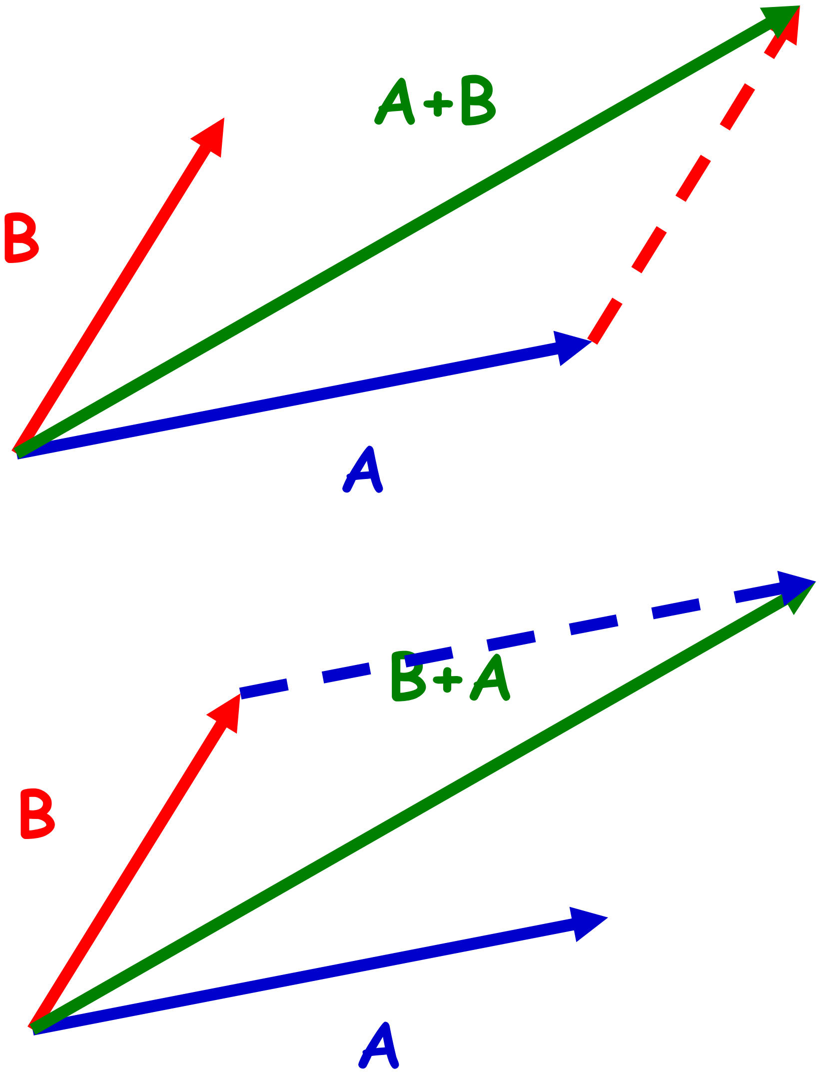
- Scalar: A
- Vector: $\vec{A}$ or $\boldsymbol{A}$

Question

- Are these two vectors the same?
- Are the lengths of these two vectors the same?

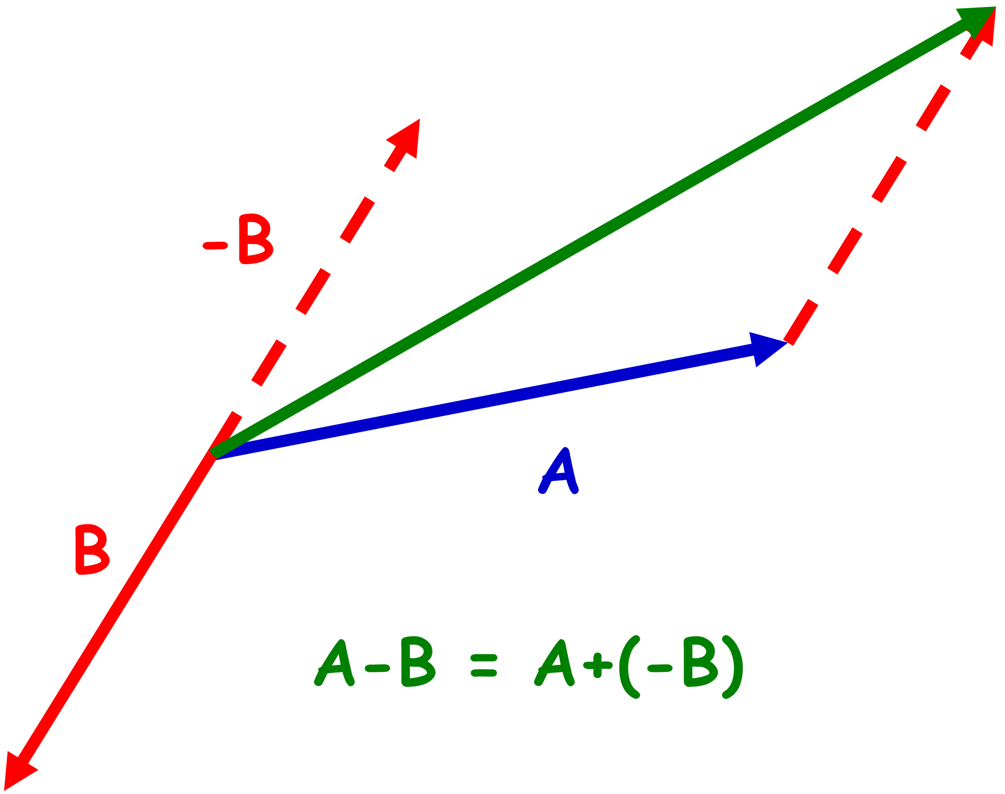
Two vectors are equal if both their length and direction are the same!

Vector addition



A+B = B+A

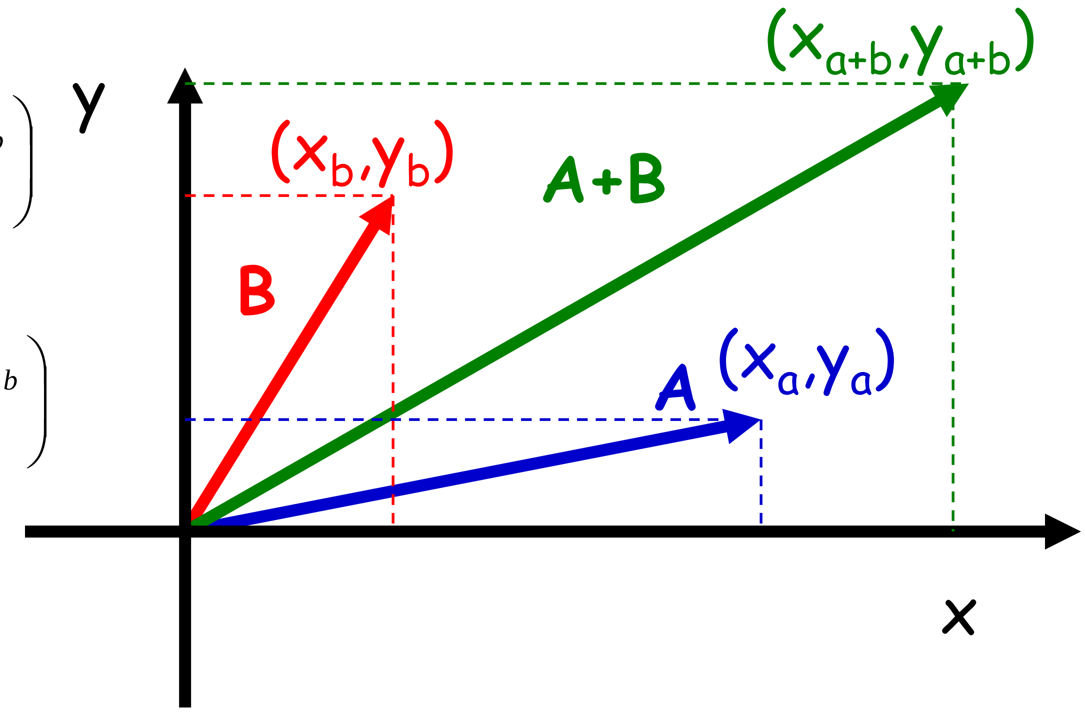
Vector subtraction



Vector operations in equations

$$\begin{pmatrix} X_{a+b} \\ Y_{a+b} \end{pmatrix} = \begin{pmatrix} X_a \\ Y_a \end{pmatrix} + \begin{pmatrix} X_b \\ Y_b \end{pmatrix} = \begin{pmatrix} X_a + X_b \\ Y_a + Y_b \end{pmatrix}$$

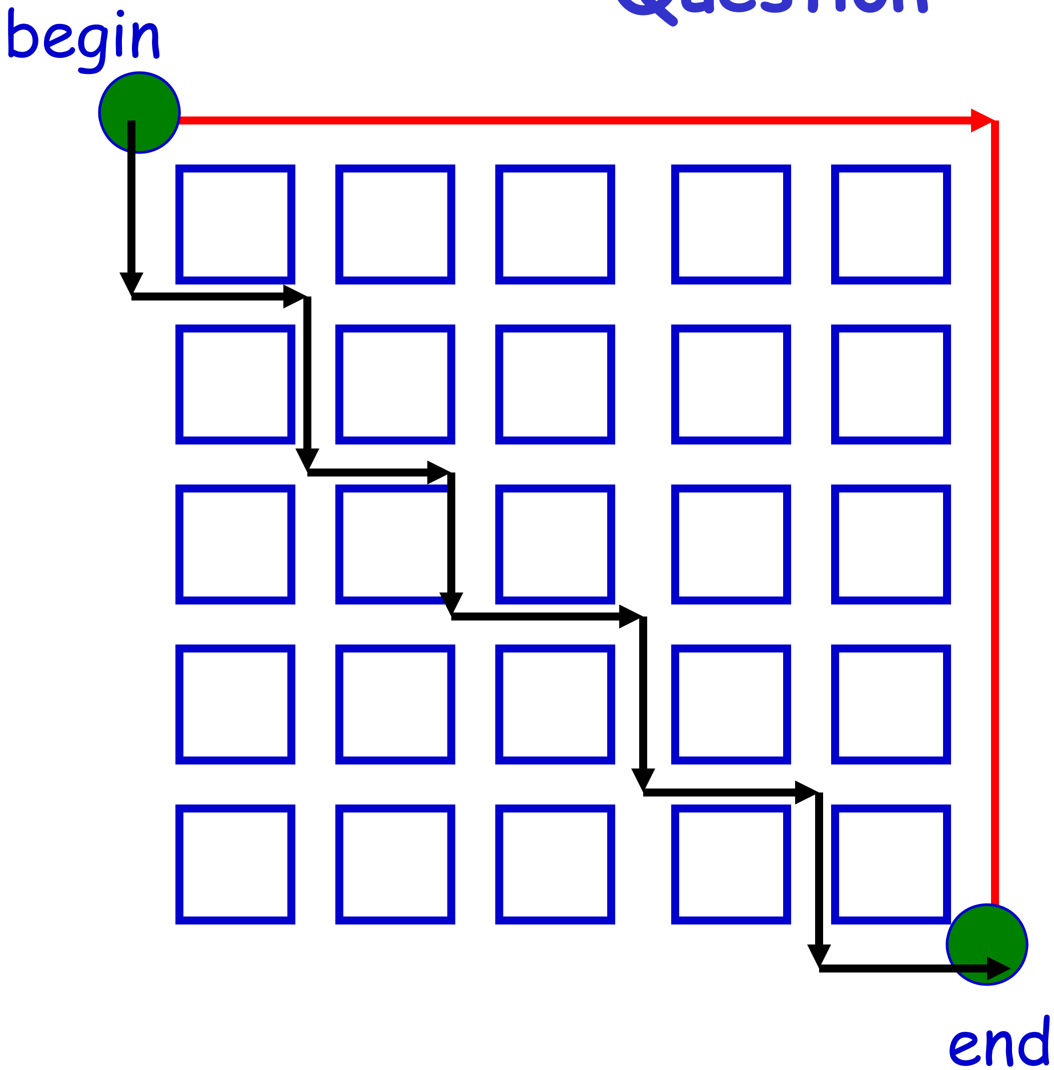
$$\begin{pmatrix} X_{a-b} \\ Y_{a-b} \end{pmatrix} = \begin{pmatrix} X_a \\ Y_a \end{pmatrix} - \begin{pmatrix} X_b \\ Y_b \end{pmatrix} = \begin{pmatrix} X_a - X_b \\ Y_a - Y_b \end{pmatrix}$$



Example:

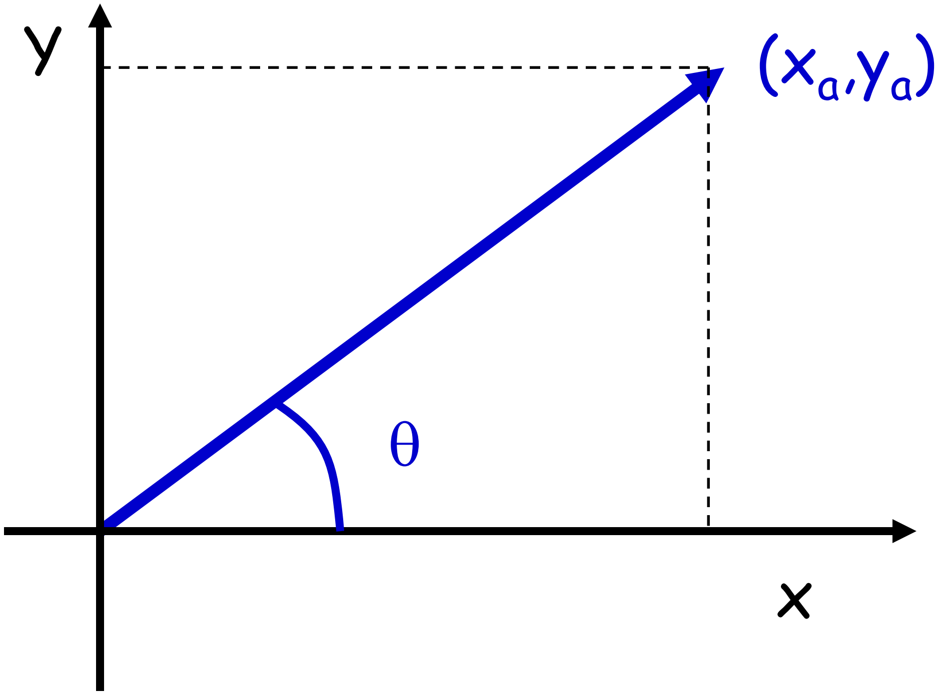
$$\begin{pmatrix} X_{a+b} \\ Y_{a+b} \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Question



Which route is shorter?

The length of a vector and its components



Length of vector (use pythagorean theorem): $l = \sqrt{x_a^2 + y_a^2}$

$$x_a = l\cos\theta$$

$$y_a = l\sin\theta$$

$$\tan\theta = y_a / x_a$$

Question

A man walks 5 km/h. He travels 12 minutes to the east, 30 minutes to the south-east and 36 minutes to the north.

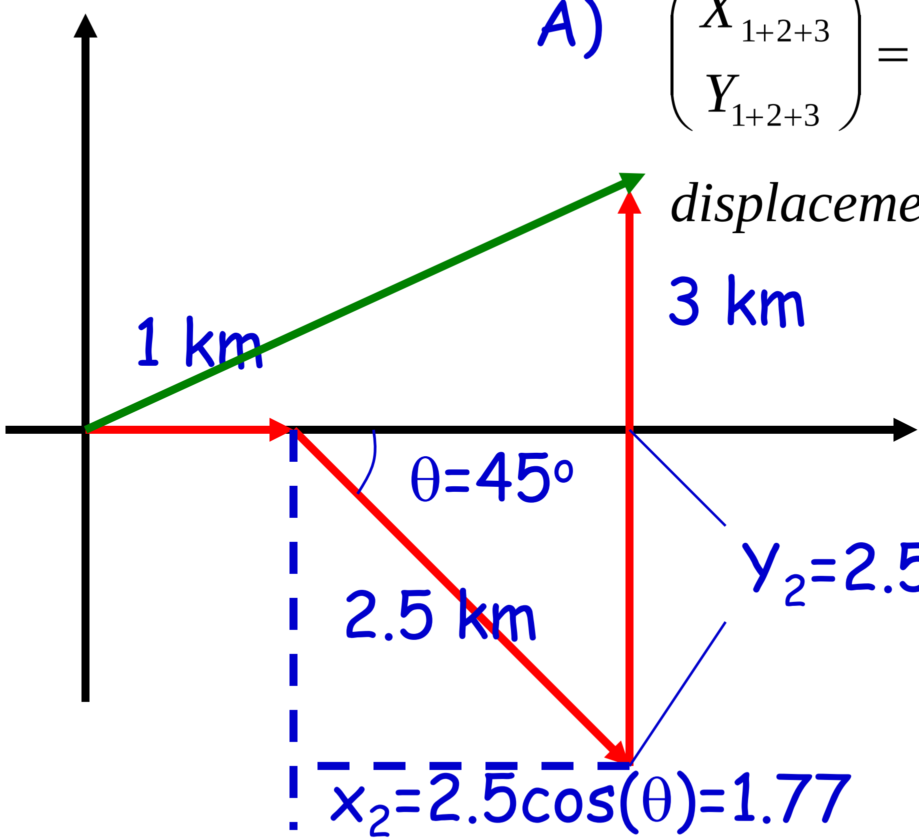
A) What is the displacement of the man?

B) What is the total distance he walked?

A)

$$\begin{pmatrix} X_{1+2+3} \\ Y_{1+2+3} \end{pmatrix} = \begin{pmatrix} X_1 + X_2 + X_3 \\ Y_1 + Y_2 + Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 1.77 + 0 \\ 0 - 1.77 + 3 \end{pmatrix} = \begin{pmatrix} 2.77 \\ 1.23 \end{pmatrix}$$

$$displacement = \sqrt{2.77^2 + 1.23^2} = 3.03$$

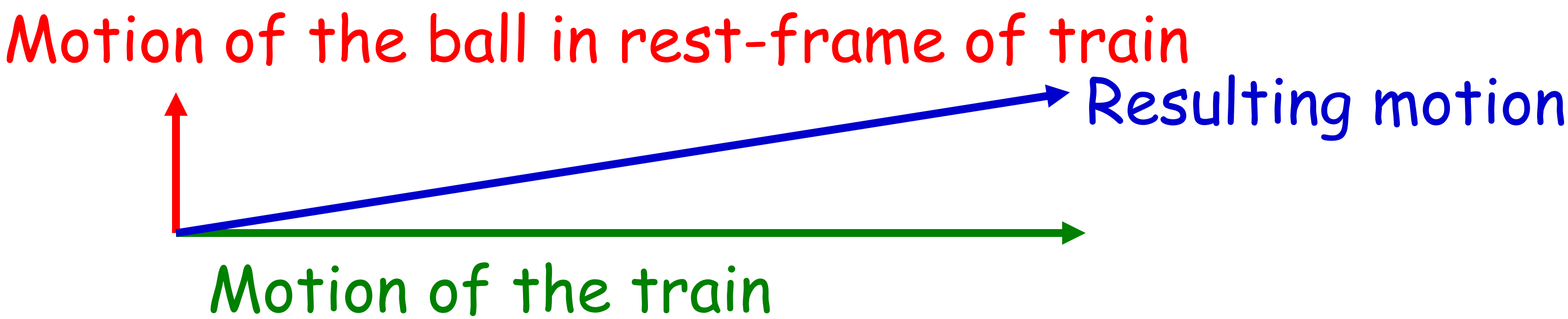


B) 1+2.5+3=6.5 km

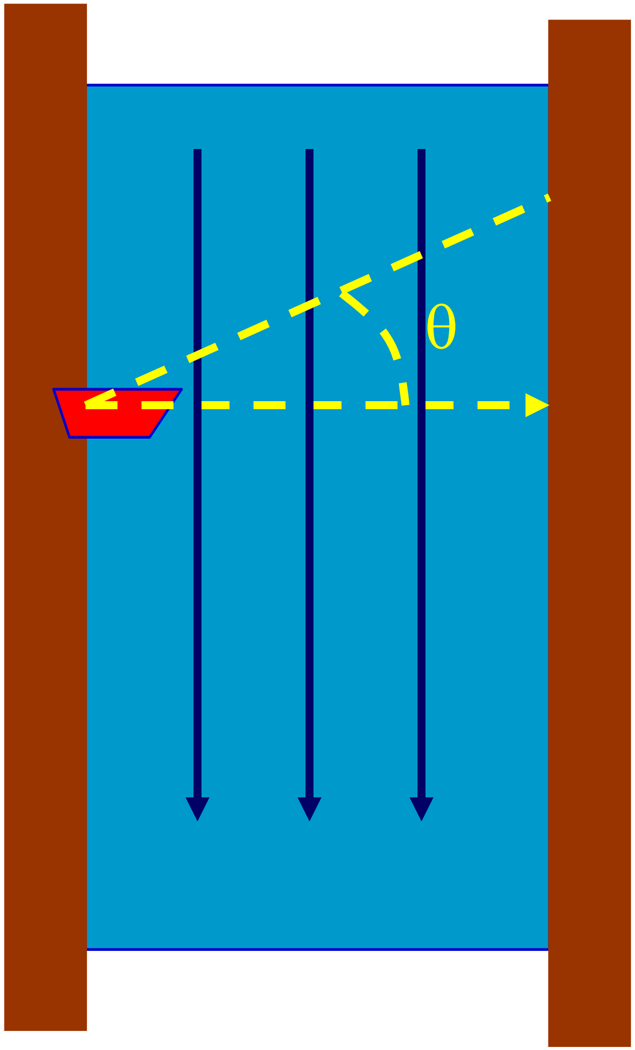
Relative motion

Motion is relative to a frame!

A woman in a train moving 50 m/s throws a ball straight up with a velocity of 5 m/s. A second person watches the train pass by and sees the woman through a window. What is the motion of the ball seen from the point of view from the man outside the train?



Question



A boat is trying to cross a 1-km wide river in the shortest way (straight across). Its maximum speed (in still water) is 10 km/h. The river is flowing with 5 km/h.

1) At what angle θ does the captain have to steer the boat to go straight across?

A) 30^o B) 45^o C) 0^o D) -45^o

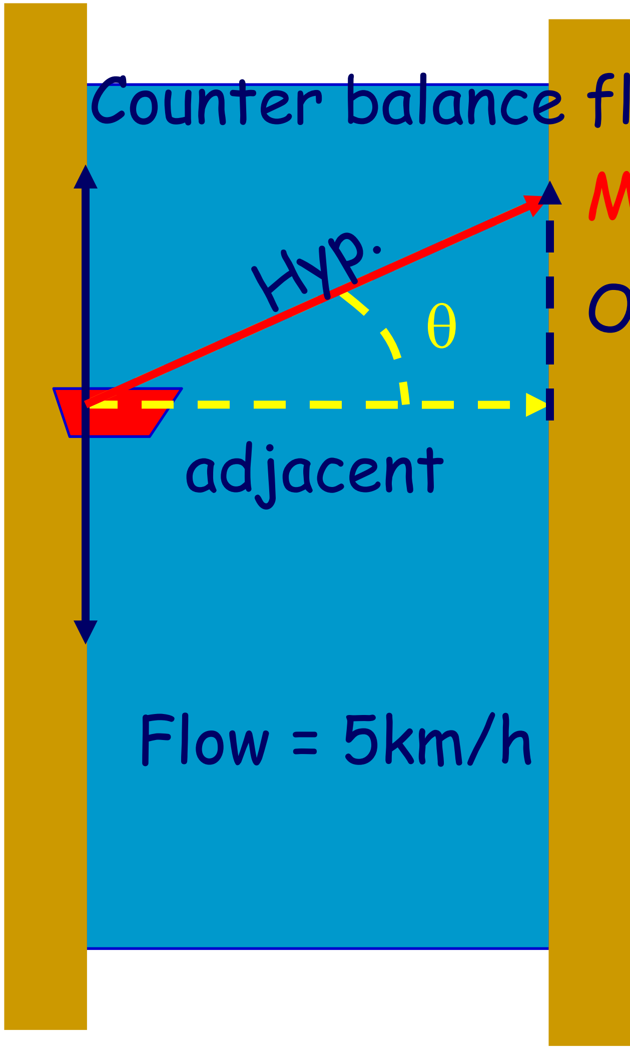
2) how long does it take for the boat to cross the river?

A) 6 min B) 6.9 min C) 12 min D) 1 h

3) If it doesn't matter at what point the boat reaches the other side, at what angle should the captain steer to cross in the fastest way?

A) 30^o B) 45^o C) 0^o D) -45^o

Answer



1) $\sin\theta$ = opposite/hypothenuse

$= 5/10 = 0.5$

$\theta = \sin^{-1} 0.5 = 30^o$

2) $\tan\theta$ = opposite/adjacent

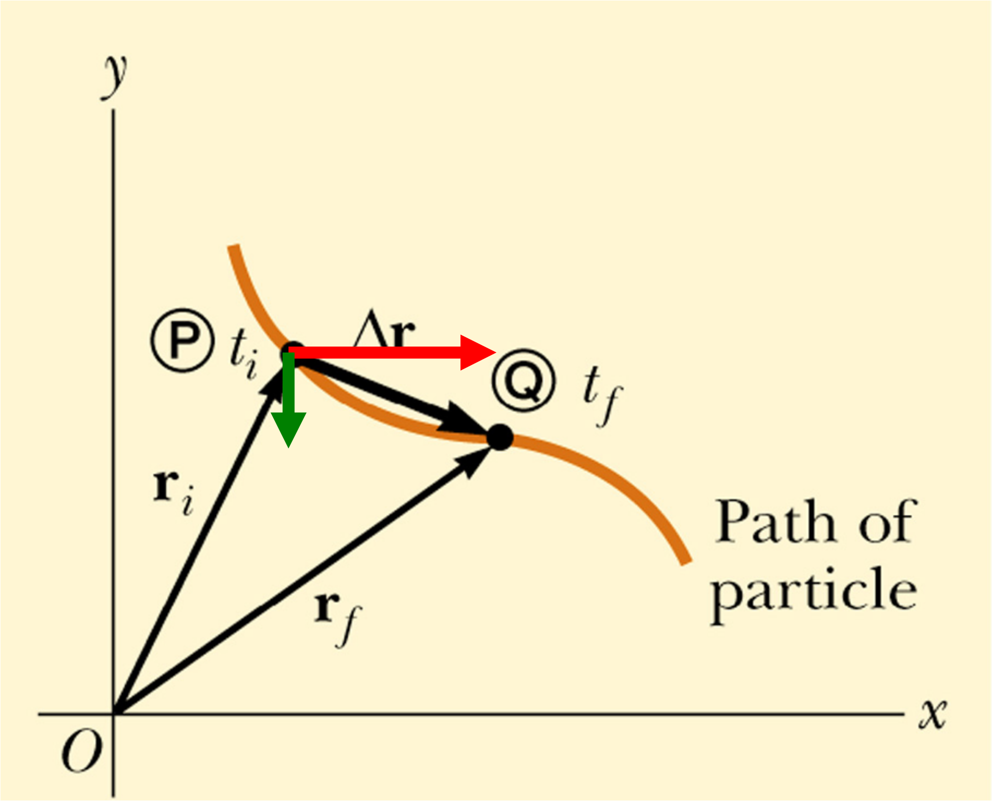
$\tan 30^o = 0.577 = 5/\text{velocity}_{hor}$

$\text{velocity}_{hor} = 8.66$ km/h

time = (1 km)/(8.66 km/h) = 0.115 h = 6.9 min

3) 0^o (the horizontal component of the velocity is then maximum.

Displacement in 2D



Often, we replace motion in 2D into horizontal and vertical components.

In vector notation:

$\Delta \mathbf{r} = \Delta \mathbf{x} + \Delta \mathbf{y}$

Velocity and acceleration

The definitions made in 1D remain the same in 2D:

$$\bar{\vec{v}} = \Delta\vec{r} / \Delta t \ldots \text{average velocity in 2D}$$

$$\vec{\mathrm{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} \ldots \text{instantaneous velocity in 2D}$$

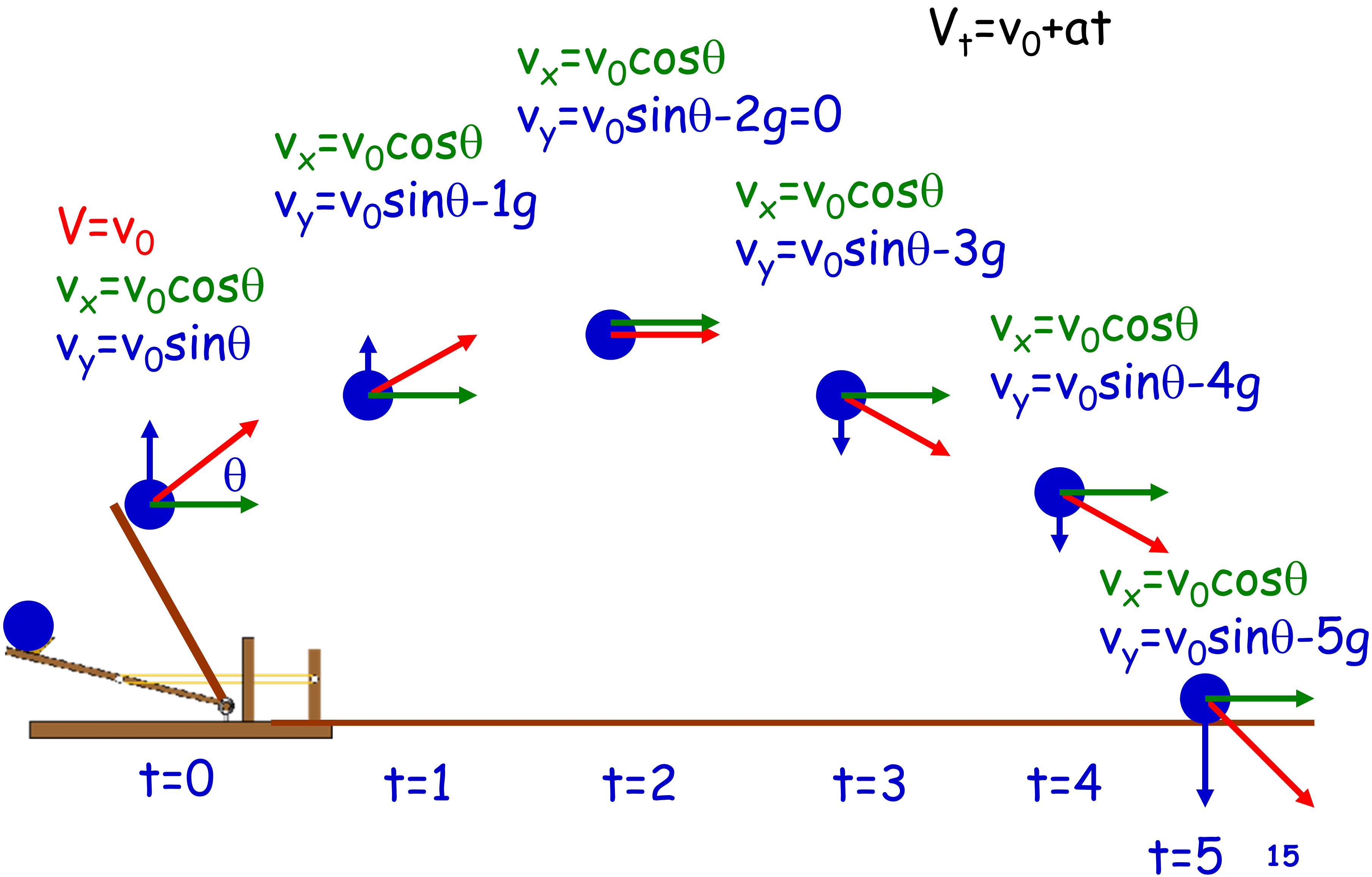
$$\bar{\vec{a}} = \Delta\vec{v} / \Delta t \ldots \text{average acceleration in 2D}$$

$$\vec{\mathrm{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} \ldots \text{instantaneous acceleration in 2D}$$

While studying motion in 2D one almost always makes a decomposition into horizontal and vertical components of the motion, which are both described in 1D

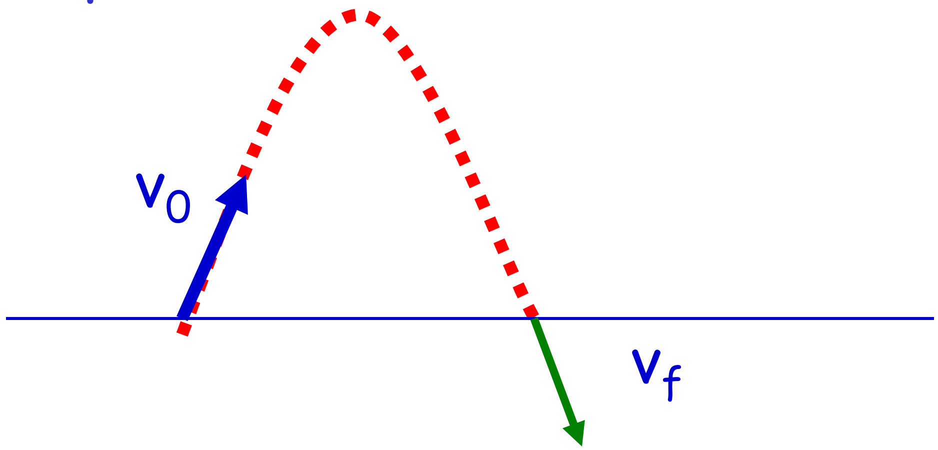
- Remember that the object can accelerate in one direction, but remains at the same speed in the other direction.
- Remember that after decomposition of 2D motion into horizontal and vertical components, you should investigate both components to understand the complete motion of a particle.
- *After decomposition into horizontal and vertical directions, treat the two directions independently.*

Parabolic motion: a catapult



Question

- A hunter aims at a bird that is some distance away and flying very high (i.e. consider the vertical position of the hunter to be 0), but he misses. If the bullet leaves the gun with a speed of v_0 and friction by air is negligible, with what speed v_f does the bullet hit the ground after completing its parabolic path?



Answer

- First consider the horizontal direction:

  $V_{0x} = V_0\cos(\theta)$

  Since there is no friction, there is no change in the horizontal component: $V_{fx} = V_0\cos(\theta) = V_{0x}$
- Next the vertical direction:

  $V_{0y} = V_0\sin(\theta)$

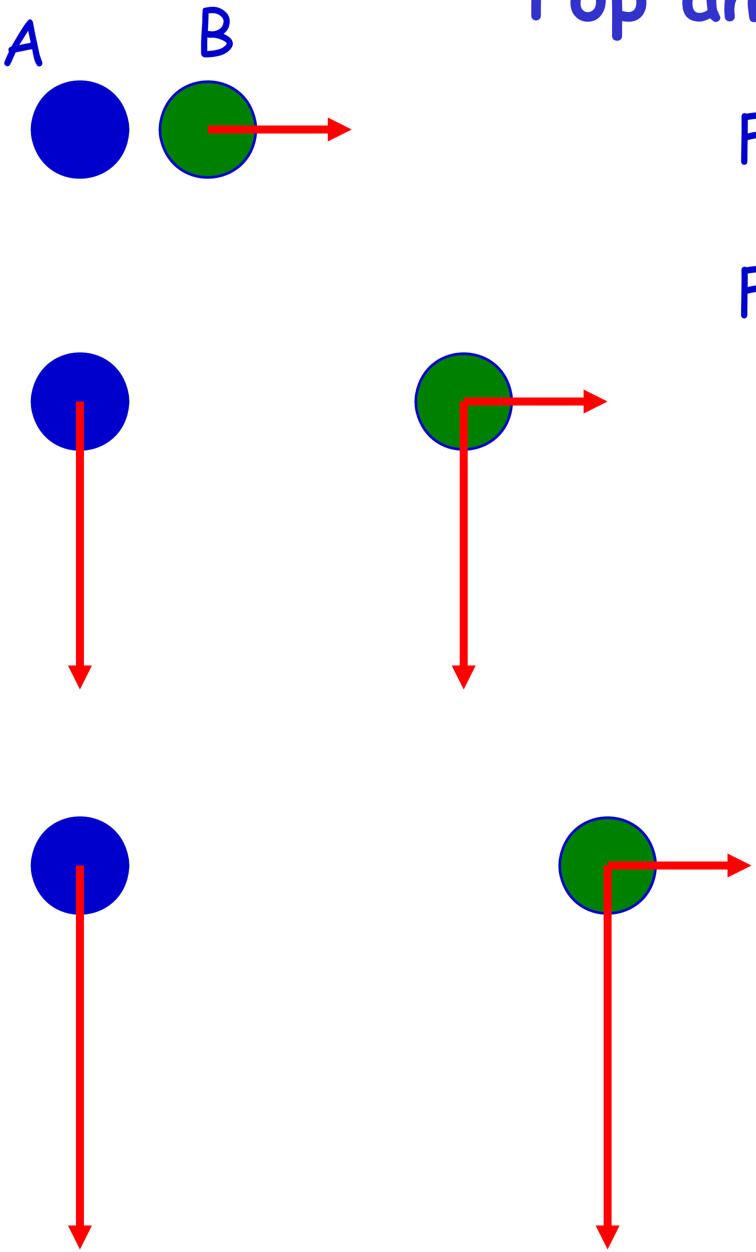
  $V_y(t) = V_{0y} - gt$ $\quad$ $x_y(t) = V_{oy}t - 0.5gt^2$ (g $=9.81\ m/s^2$)

  Boundary condition: bullet hits the ground:

  $0 = V_{oy}t - 0.5gt^2 \longrightarrow$ $\quad t = 0$ or $t = 2V_{0y}/g$

  So, $V_{fy}(t) = V_{0y} - (2V_{0y}/g)g = -V_{0y}$
- Total velocity = $[V_{0x}{}^2 + (-V_{0y})^2]^{1/2} = V_0$!!!!
- The speed of the bullet has not changed, but the vertical component of the velocity has changed sign.

Pop and Drop



For A: $V_y = -0.5gt^2$

$V_x = 0$

For B: $V_y = -0.5gt^2$

$V_x = V_0$

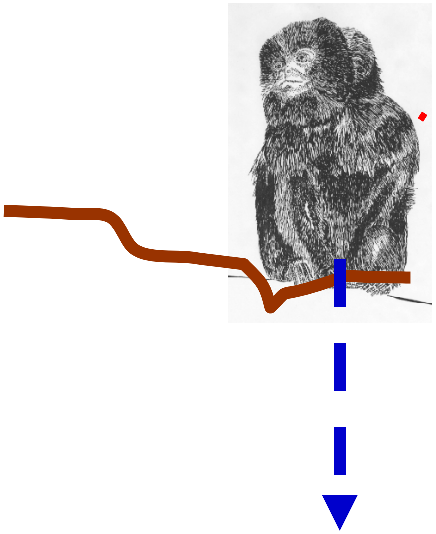
For A: $X_y = X_0 - 0.5gt^2$

$X_x = 0$

For B: $X_y = X_0 - 0.5gt^2$

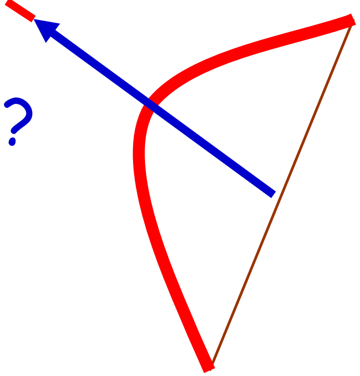
$X_x = V_0 t$

Shoot the monkey



The hunter aims his arrow exactly at the monkey

At the moment he fires, the monkey drops off the branch. What happens?



The hor. position of the arrow is:

$x(t) = d - v_0\cos(\theta)t$

$x(t) = 0$ at $t = d/v_0\cos(\theta) = t_{x=0}$

Its vertical position is:

$y(t) = v_0\sin(\theta)t - 0.5gt^2$

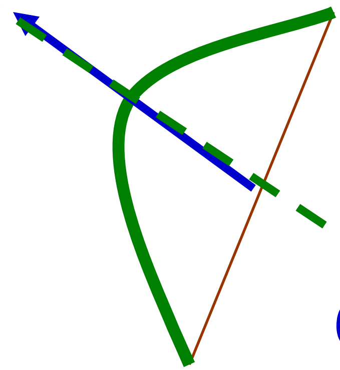
$y(t_{x=0}) = d\tan(\theta) - 0.5gt^2 = h - 0.5gt^2$

Always hit!!

h

The vertical position of the monkey is: $y(t) = h - 0.5gt^2$

The horizontal position is 0



d

θ

Another example

- A football player throws a ball with initial velocity of 30 m/s at an angle of 30° degrees w.r.t. the ground. How far will the ball fly before hitting the ground? And what about 60°? And at what angle is the distance thrown maximum?

$X(t) = 30\cos(\theta)t$

$Y(t) = 30\sin(\theta)t - 0.5gt^2$

$= 0$ if $t(30\sin(\theta) - 0.5gt) = 0$

$t = 0$ or $t = 30\sin(\theta)/(0.5g)$

$X(t = 30\sin(\theta)/(0.5g)) = 900\cos(\theta)\sin(\theta)/(0.5g)$

$= 900\sin(2\theta)/g$

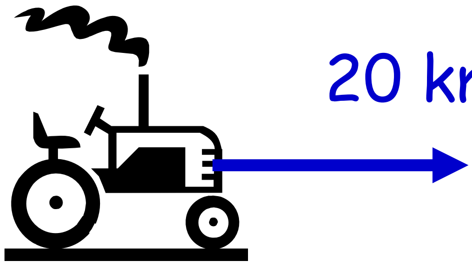
if $\theta = 30^o$ $X = 79.5$ m

if $\theta = 60^o$ $X = 79.5$ m !!

Maximum if $\sin(2\theta)$ is maximum, so $\theta = 45^o$

$X(\theta = 45^o) = 91.7$ m

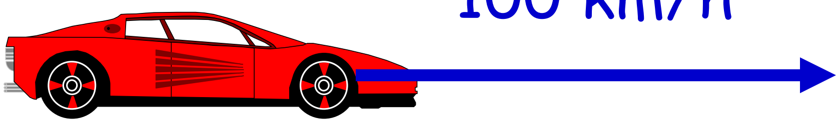
Relative motion of 2 objects



20 km/h

What is the velocity of
the Ferrari relative to the tractor?
And the other way around?

100 km/h



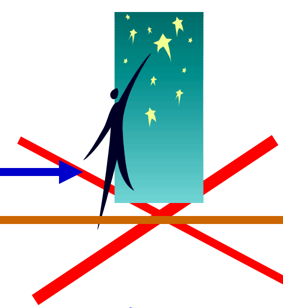
A) 80 km/h & 80 km/h
B) 20 km/h & -80 km/h
C) 80 km/h & -80 km/h
D) 100 km/h & 20 km/h

Relative motion of 2 objects II

A UN plane drops a food package from a distance of 500 m high aiming at the dropzone X.
What does the motion of the package look like from the point of view of
a) the pilot   b) the people at the drop zone

500m

d



Recall of previous Lecture: if the plane is going at 100m/s, at what distance d from X should the plane drop the package?

Answer

Horizontal direction: $x(t) = x_0 + v_0 t + 0.5at^2$

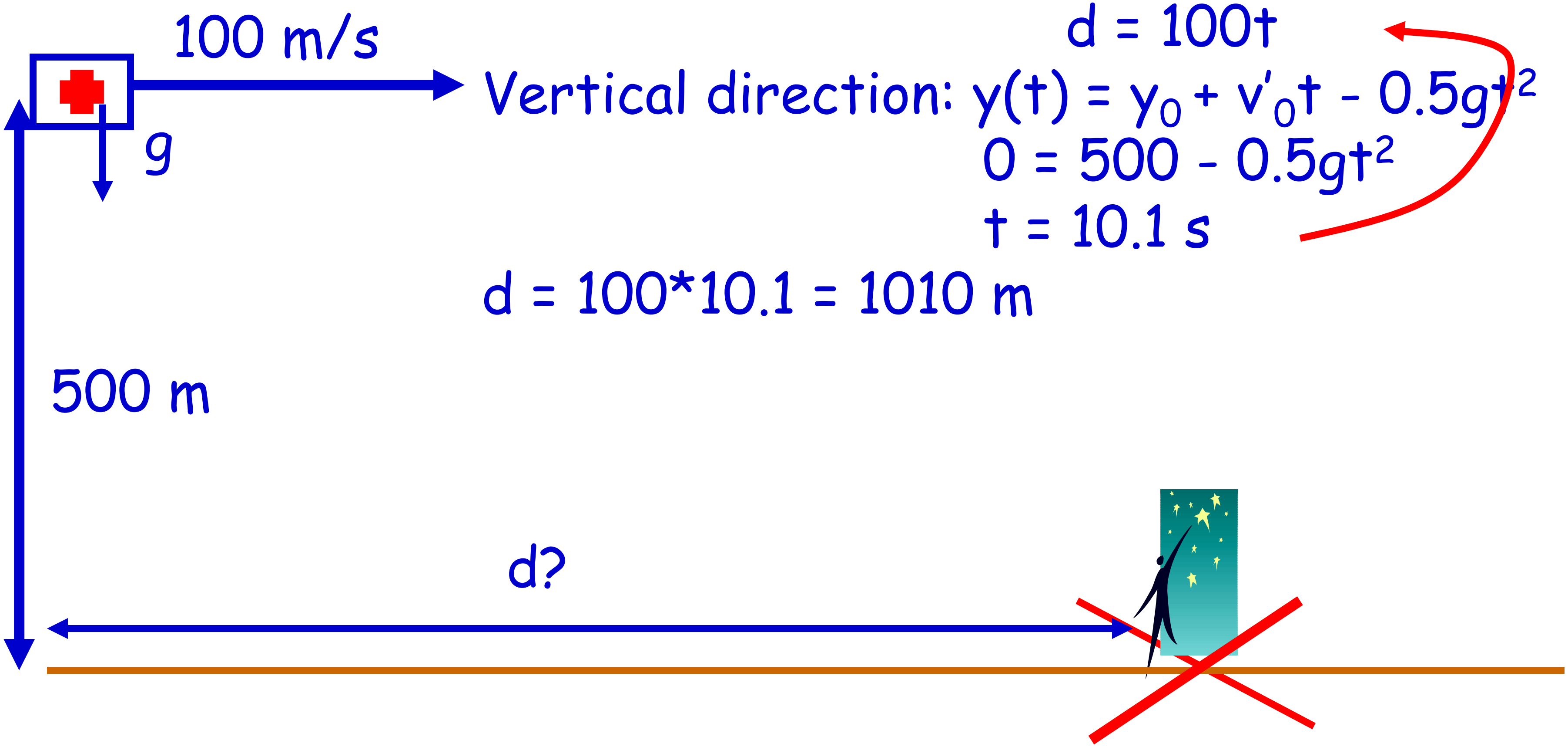
$d = 100t$

Vertical direction: $y(t) = y_0 + v'_0 t - 0.5gt^2$

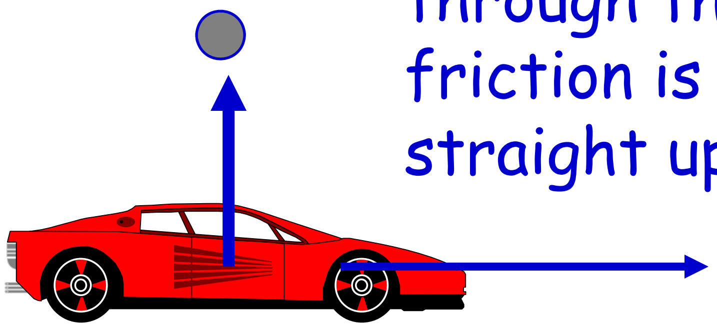
$0 = 500 - 0.5gt^2$

$t = 10.1\ s$

$d = 100*10.1 = 1010\ m$



A careless driver.



A man driving in his sportscar finishes his drink and throws the can out of his car through the sun roof. Assuming that air friction is negligible and his throw is straight up, what happens?

For the can: horizontal direction: $x(t) = v_{car}t$

vertical direction: $y(t) = v_{drink}t - 0.5gt^2 = 0$ if

$t = 0$ (start) or $t = (2V_{drink}/g)^{1/2}$

At $t = (2V_{drink}/g)^{1/2}$

For the car: horizontal direction: $x(t) = v_{car}t$

After $t = (2V_{drink}/g)^{1/2}$ the can drops back on the drivers head!

Range

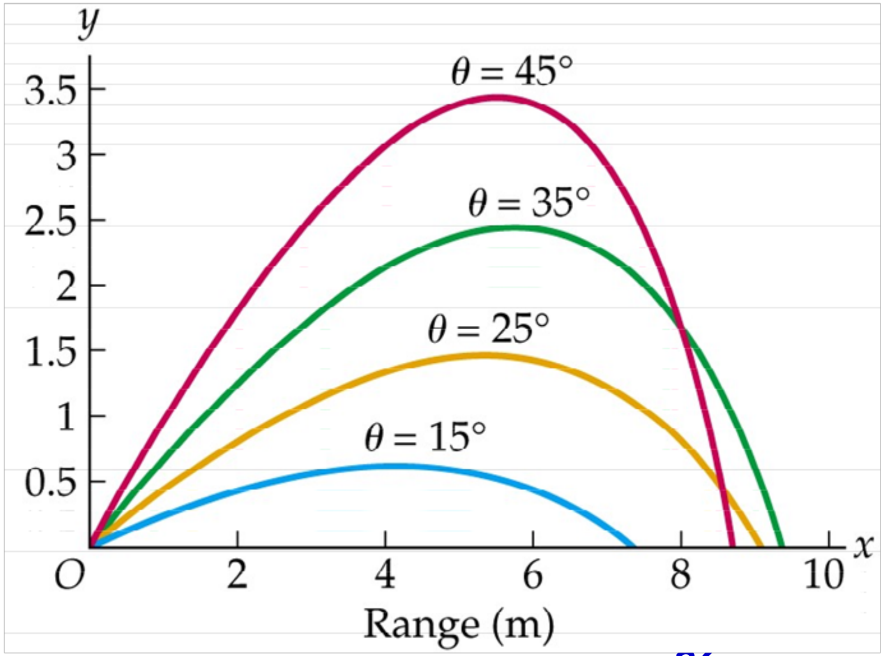
The range R of a projectile is the horizontal distance it travels before landing.

$$R = \left(\frac{v_0^2}{g}\right)\sin 2\theta$$
$$= \left[2v_{0,x}v_{0,y}\right]/\,g$$

assuming same initial and final elevation

What angle θ results in the maximum range?

What if we do not ignore air resistance?



Range (unequal heights)

$$y = y_0 + v_{0,y}t + \frac{1}{2}at^2$$

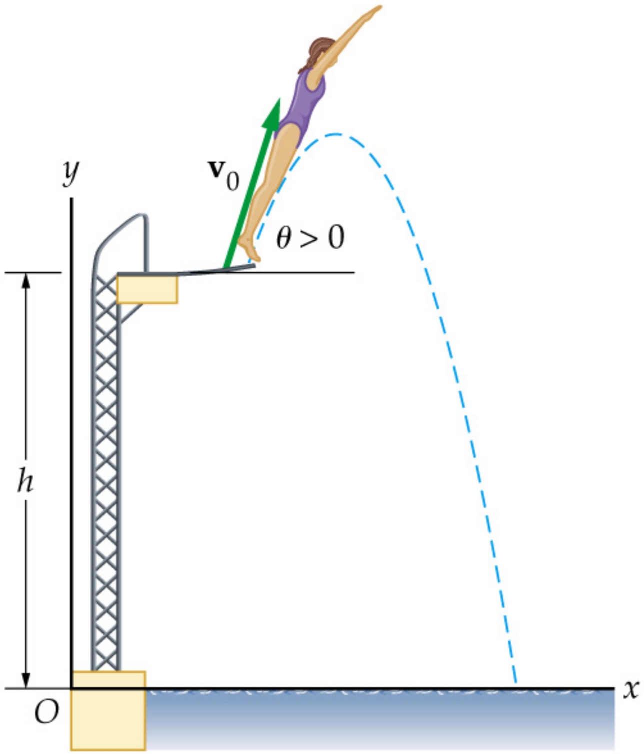
$$x = x_0 + v_{0,x}t$$

$$x_0 = 0, \qquad y_0 = h$$

$$v_{0,x} = v_0 \cos\theta, \qquad v_{0,y} = v_0 \sin\theta$$

$$a = -g$$

$$y = h + [v_0 \sin\theta]t - \frac{1}{2}gt^2$$



Set y=0 and solve quadratic for t

$$t = \frac{-v_0 \sin\theta \pm \sqrt{[v_0 \sin\theta]^2 - 4\left[-\frac{g}{2}\right]h}}{-g} = \frac{v_0 \sin\theta}{g}\left[1 \mp \sqrt{1 + \frac{2gh}{(v_0 \sin\theta)^2}}\right] \Longrightarrow \quad Range = v_{0,x}t$$

Maximum Height

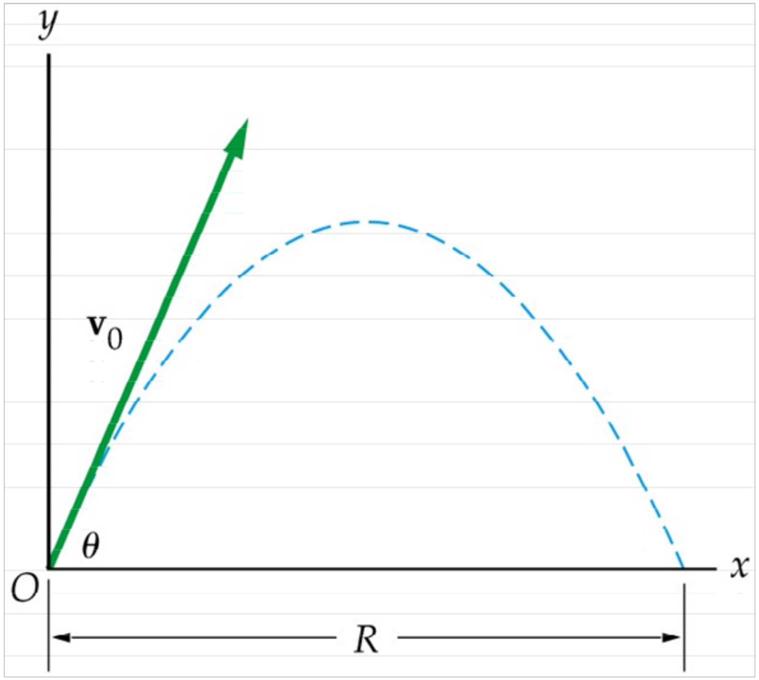
The maximum height (and therefore the "hang time") of a projectile depends only on the vertical component of its initial velocity.

At y_{max}, the vertical velocity v_y is zero.

$$v_y^2 = v_{0y}^2 + 2a\Delta y$$

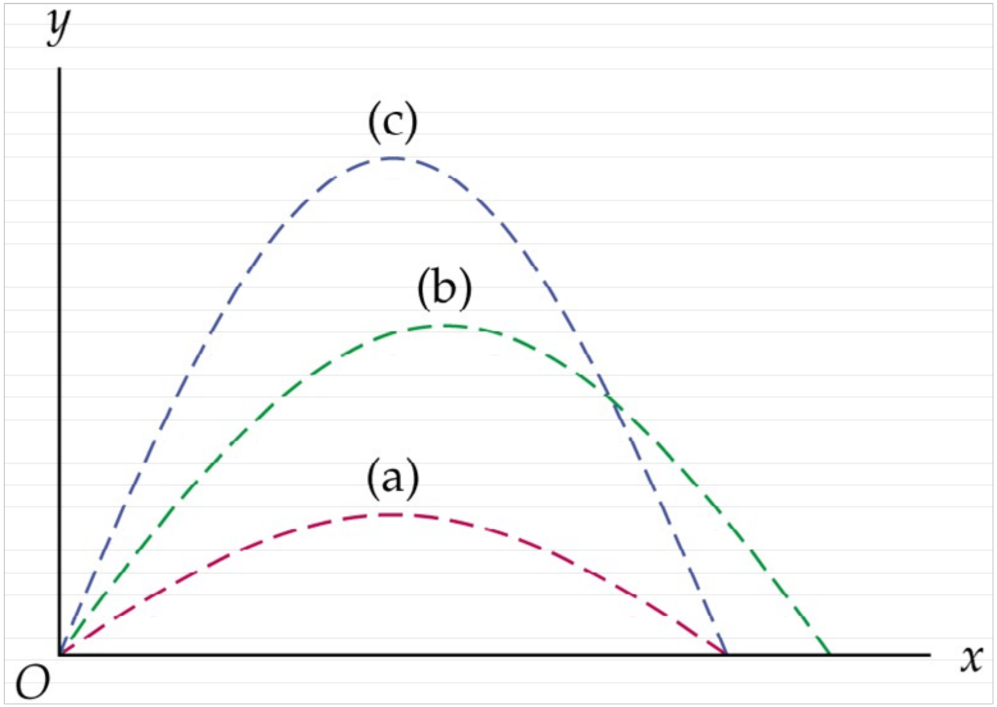
$$0 = v_0^2 \sin^2\theta + 2(-g)y_{\max}$$

$$y_{\max} = \frac{v_0^2 \sin^2\theta}{2g}$$



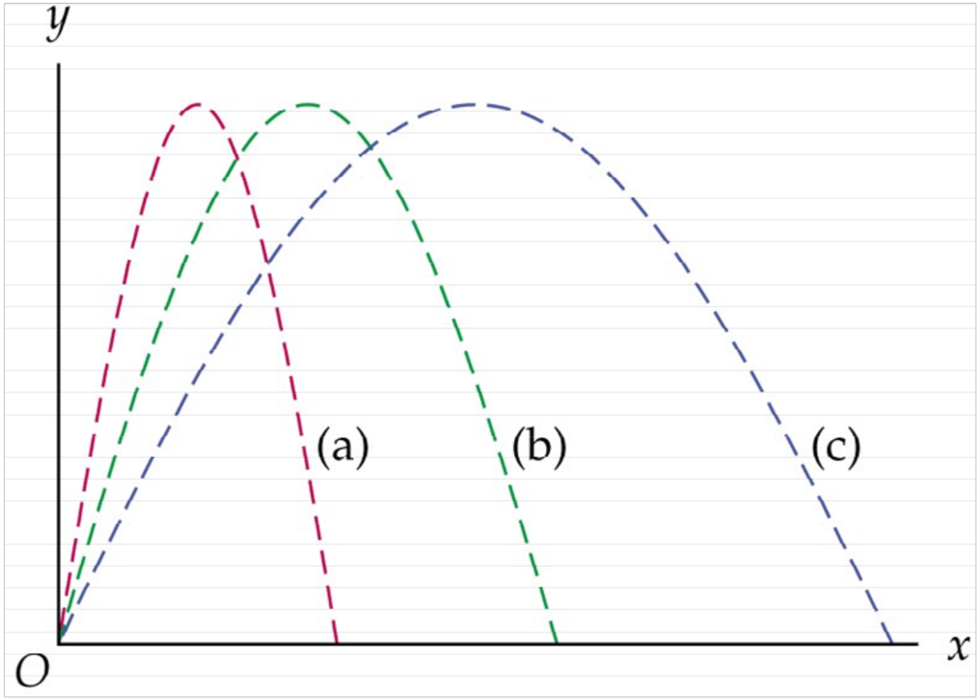
Problem

Three projectiles (a, b and c) are launched with the same initial speed but with different launch angles, as shown. List the projectiles in order of increasing (a) horizontal component of initial velocity and (b) time in flight



Problem

Three projectiles (a, b and c) are launched with different initial speeds so that they reach the same maximum height, as shown. List the projectiles in order of increasing (a) initial speed and (b) time of flight.



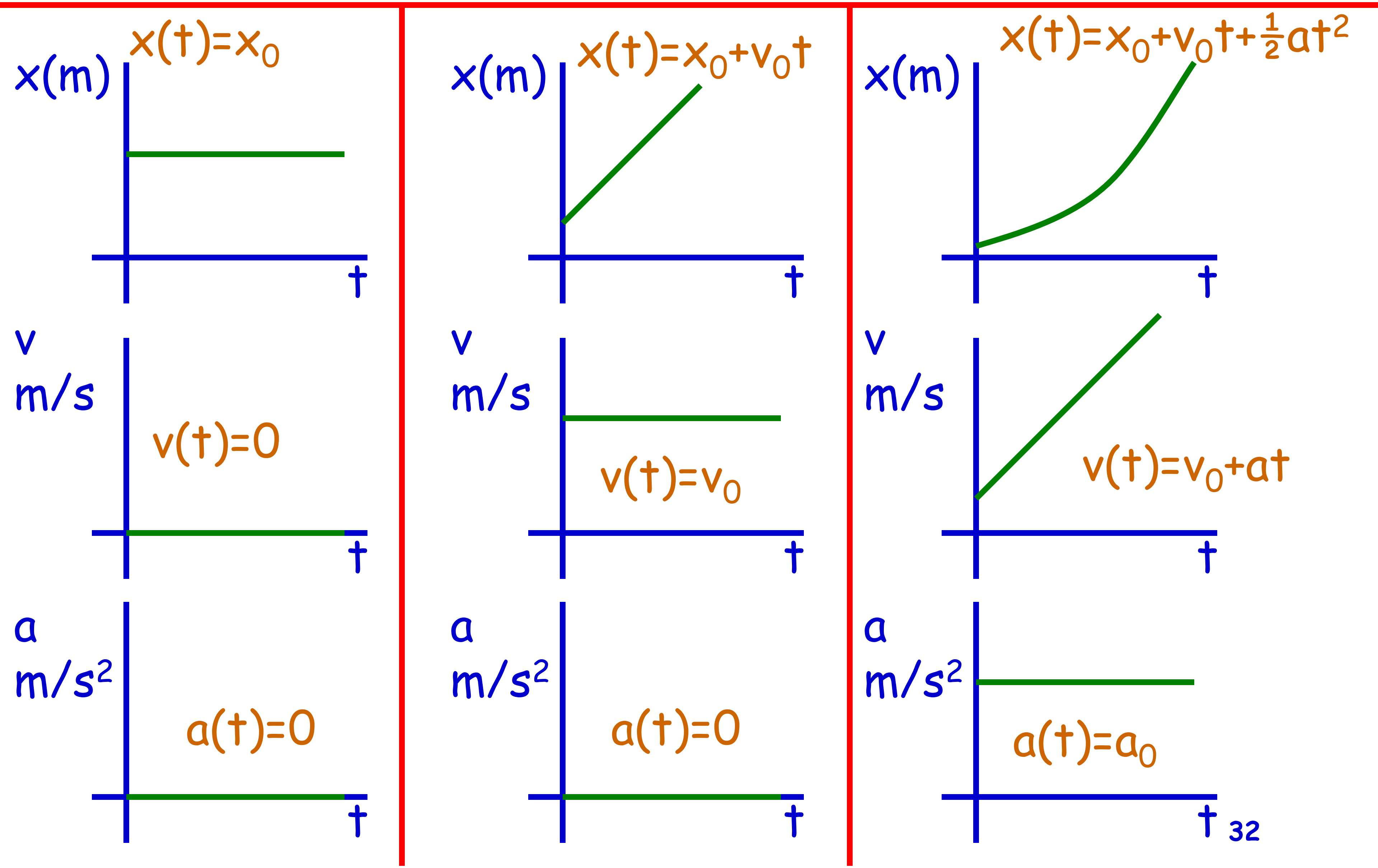
Problem

Driving down the highway you find yourself behind a heavily loaded tomato truck. You follow close behind the truck, keeping the same speed. Suddenly a tomato falls from the back of the truck. Will the tomato hit your car or land on the road, assuming you continue moving with the same speed and direction?

(Neglect air friction)

Important things!

Constant motion | Constant velocity | Constant acceleration



About signs:

- Distance, velocity and acceleration have signs (vectors)
- If its velocity is negative, an object is moving in the negative direction ($x(t) = x_0 - |v|t$)
- If its acceleration is positive, an object is increasing velocity (making it more positive or less negative)
- If its acceleration is negative, an object is decreasing velocity (making it less positive or more negative)
- To keep your signs in check, choose a coordinate system and stick to it when solving the problem.

- Before trying to solve an equation numerically, make a sketch of the motion using the motion diagrams in the previous page.

2D motion

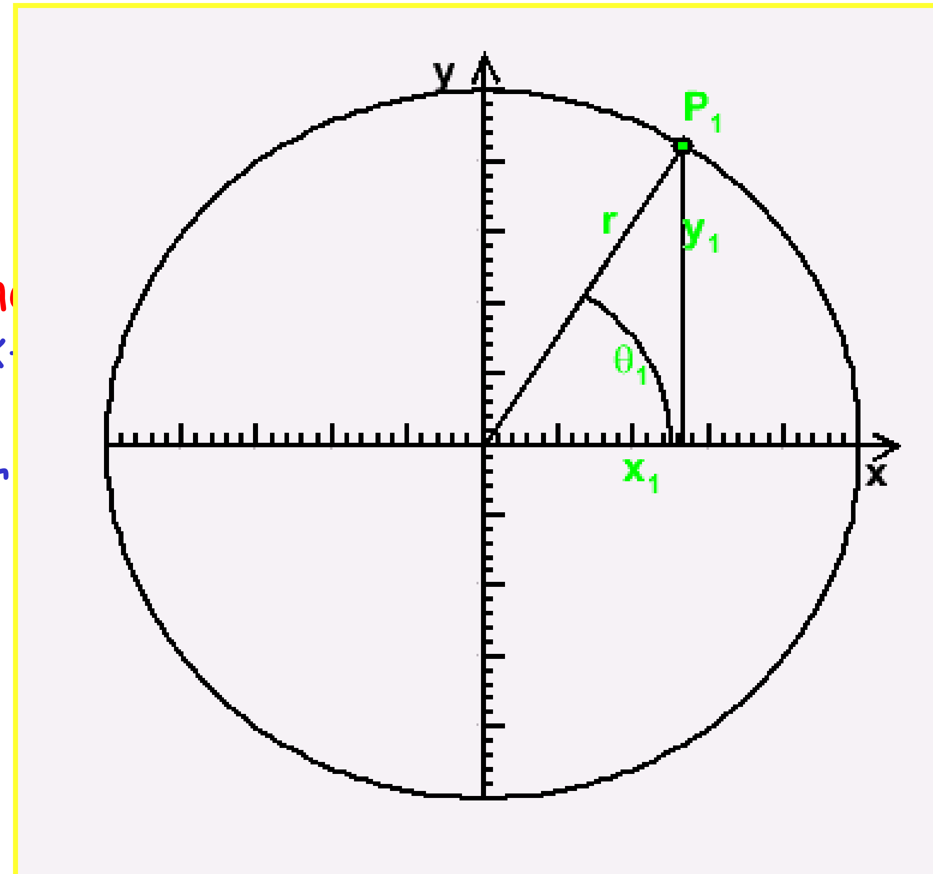
- When trying to understand the motion of an object in 2D decompose the motion into vertical and horizontal components.
- Be sure of your coordinate system; is the motion of the object you want to study relative to another object?
- Write down the equations of motion for each direction separately.
- If you cannot understand the problem, draw motion diagrams for each of the directions separately.
- Make sure you understand which quantity is unknown, and plug in the equation of motions the quantities that you know (given). Then solve the equations.

APPENDIX:
Trigonometry and Vector Components

- Trigonometry is a pre-requisite for this course.
- Now you will learn ½ of trigonometry, and most part of what you need for this course.
- In this discussion, we always define the direction of a vector in terms of an angle counter-clockwise from the + x-axis.
- Negative angles are measured clockwise.

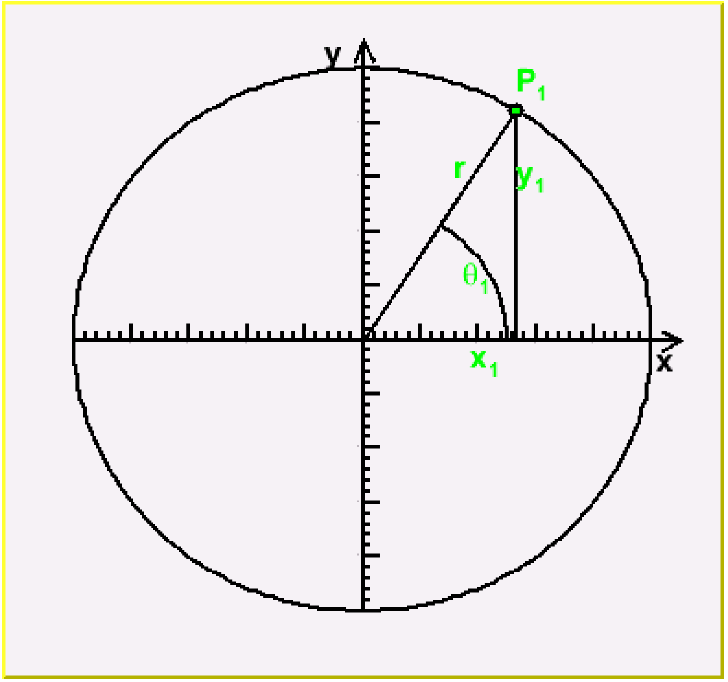
Trigonometry and Circles

- The point lies on a circle of radius r.
- The line from the origin to makes an angle w.r.t. the x-axis.
- The trigonometric functions sin and cosine are defined by the x- and y-components of P_1:
  - $x_1 = r \cos(\theta_1)$: $\cos(\theta_1) = x_1 / r$
  - $y_1 = r \sin(\theta_1)$: $\sin(\theta_1) = y_1 / r$
  - Tangent of $(\theta_1) = y_1 / x_1$
  - $\tan(\theta_1) = [\sin(\theta_1)] / [\cos(\theta_1)]$



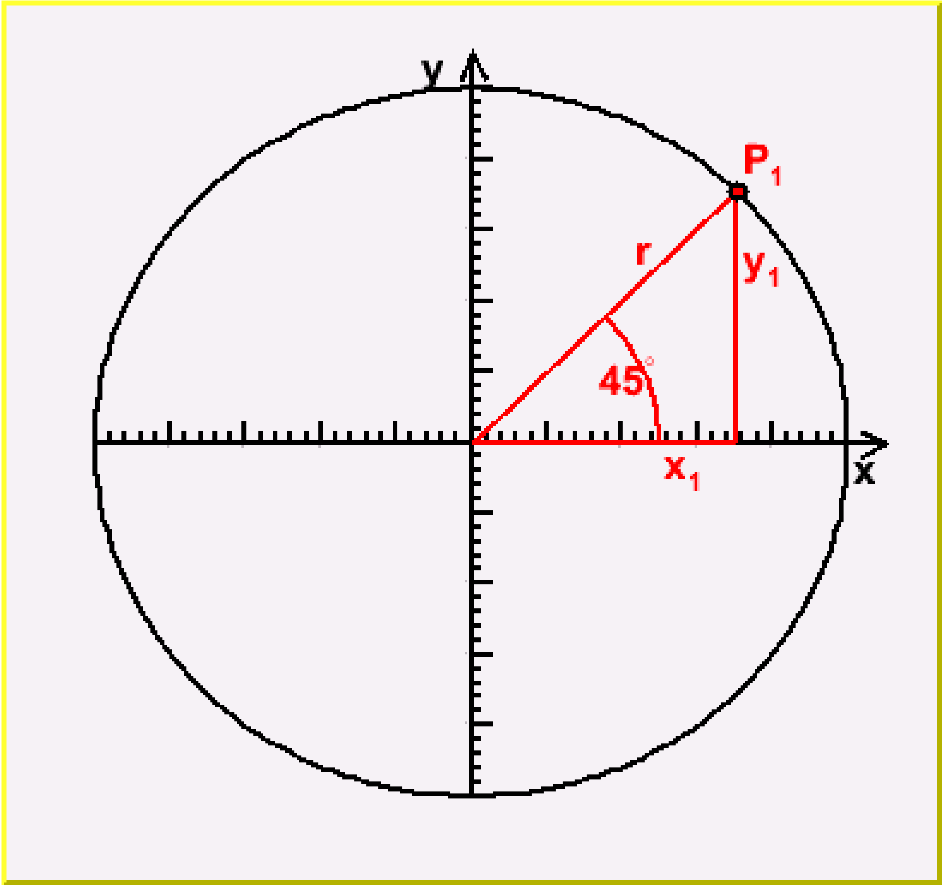
Special (simple) cases of sine and cosine

- $\cos(0°) = 1$, $\sin(0°) = 0$
- $\cos(90°) = 0$, $\sin(90°) = 1$
- $\cos(180°) = -1$, $\sin(180°) = 0$
- $\cos(270°) = 0$, $\sin(270°) = -1$
- Sine and Cosine are periodic functions:
  - $\cos(\theta+360) = \cos(\theta)$
  - $\sin(\theta+360°) = \sin(\theta)$



45-45-90 triangle

- By symmetry,
  - $x_1 = y_1$
- Pythagoras:
  - $x_1^2 + y_1^2 = r^2$
  - $2 \cdot x_1^2 = r^2$
  - $x_1 = r/\sqrt{2}$
- $\cos(45°) = x_1 / r = 1/\sqrt{2}$
- $\cos(45°) = 0.7071...$
- $\mathrm{Sin}(45°) = 1/\sqrt{2}$



30-60-90 Triangle

- From Equilateral triangle:
  - $2 \cdot y_1 = r$
- Pythagoras:

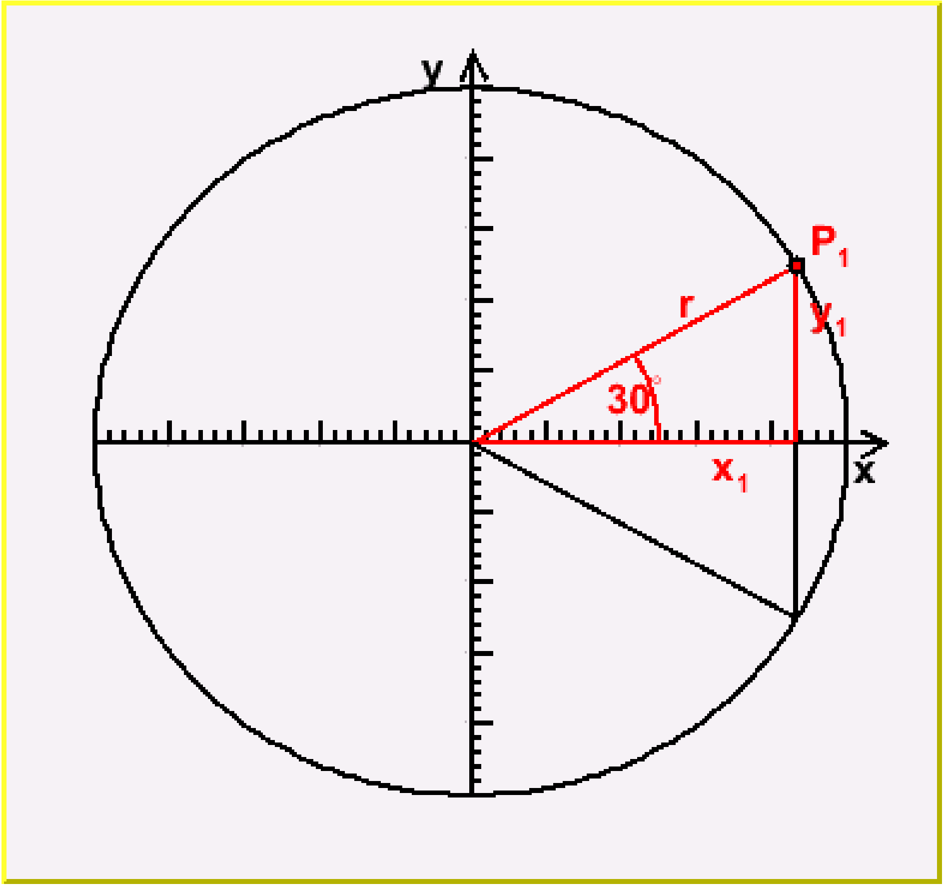
$$x_1^2 + y_1^2 = r^2$$

$$x_1^2 + (r/2)^2 = r^2$$

$$x_1^2 = \frac{3}{4} r^2$$

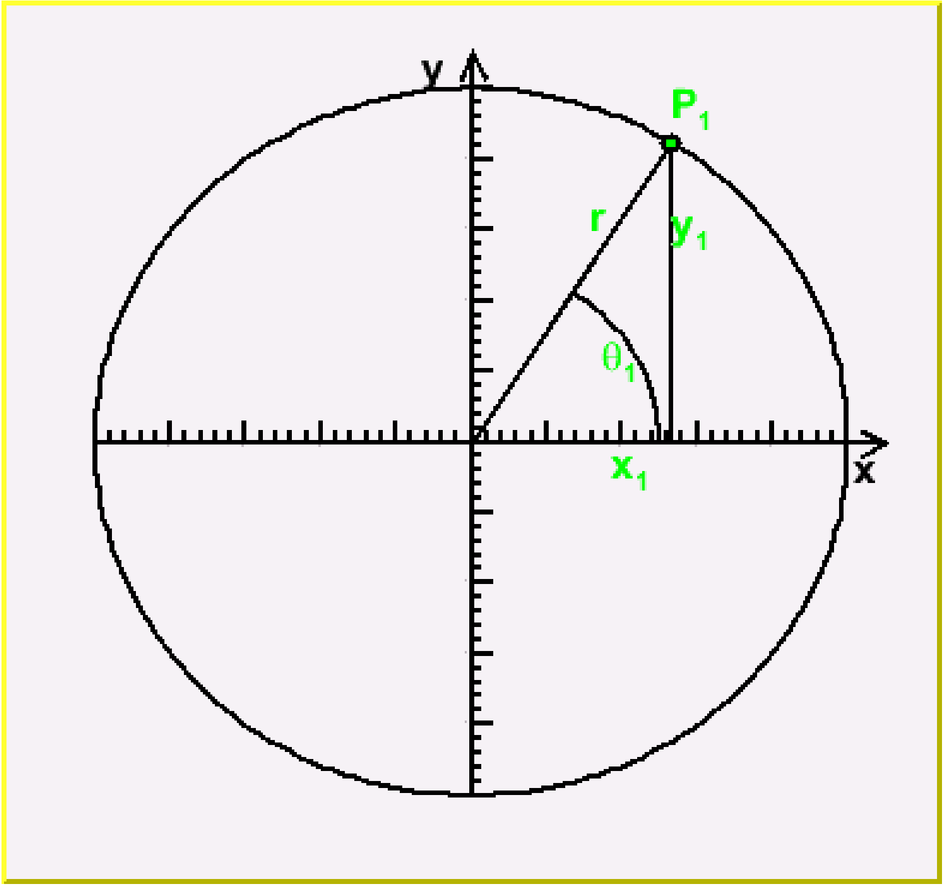
$$\cos 30° = \frac{x_1}{r} = \frac{\sqrt{3}}{2} = 0.866\ldots$$

$$\sin 30° = \frac{y_1}{r} = \frac{1}{2}$$



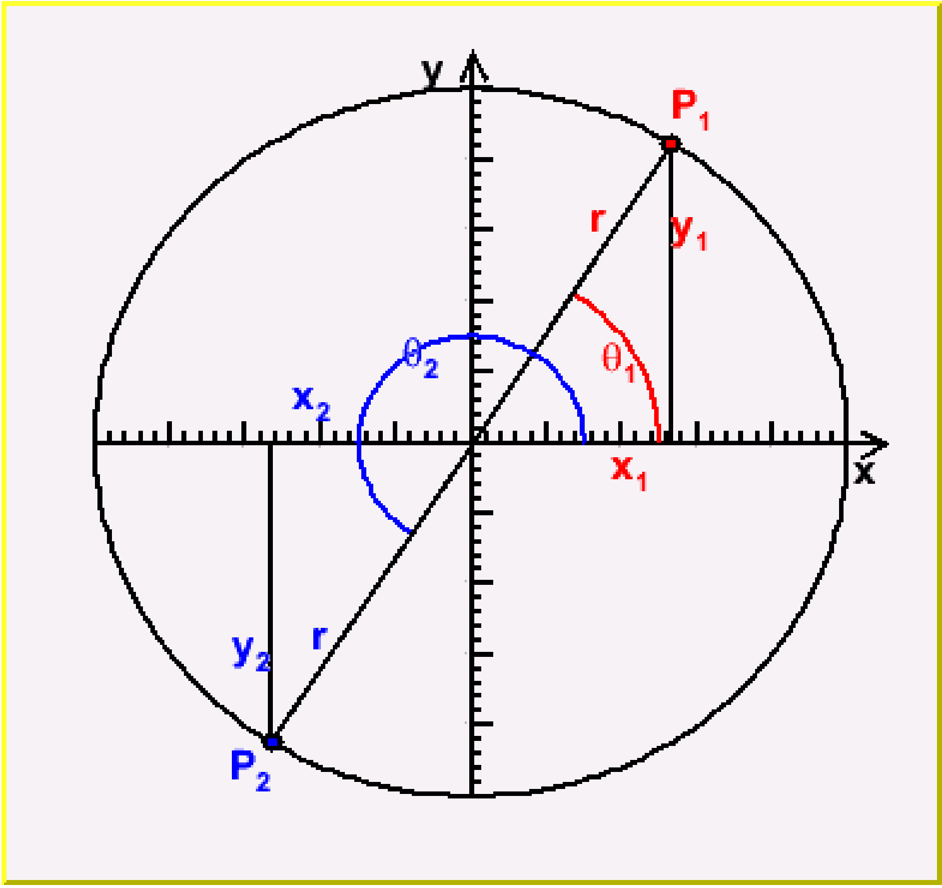
Navigating the Quadrants (Circles are better than Triangles)

- First Quadrant:
  - $0° < \theta < 90°$
  - $\cos(\theta) > 0$, $\sin(\theta) > 0$
- Second Quadrant
  - $90° < \theta < 180°$
  - $\cos(\theta) < 0$, $\sin(\theta) > 0$
- Third Quadrant
  - $180° < \theta < 270°$
  - $\cos(\theta) < 0$, $\sin(\theta) < 0$
- Forth Quadrant
  - $0° < \theta < 90°$
  - $\cos(\theta) > 0$, $\sin(\theta) < 0$



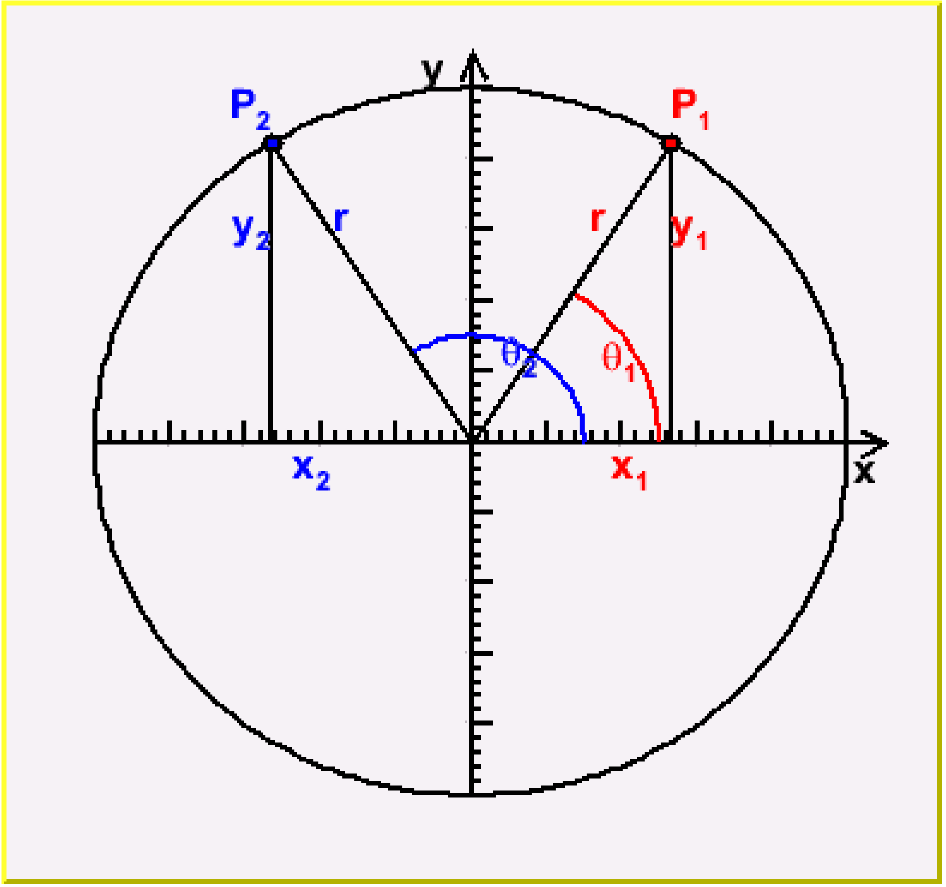
Moving from Quadrant to Quadrant: Adding 180 degrees

- $\theta_2 = \theta_1 + 180°$
- $x_2 = -x_1$, $y_2 = -y_1$
- $\cos(\theta_1 + 180°) = -\cos(\theta_1)$
- $\sin(\theta_1 + 180°) = -\sin(\theta_1)$.



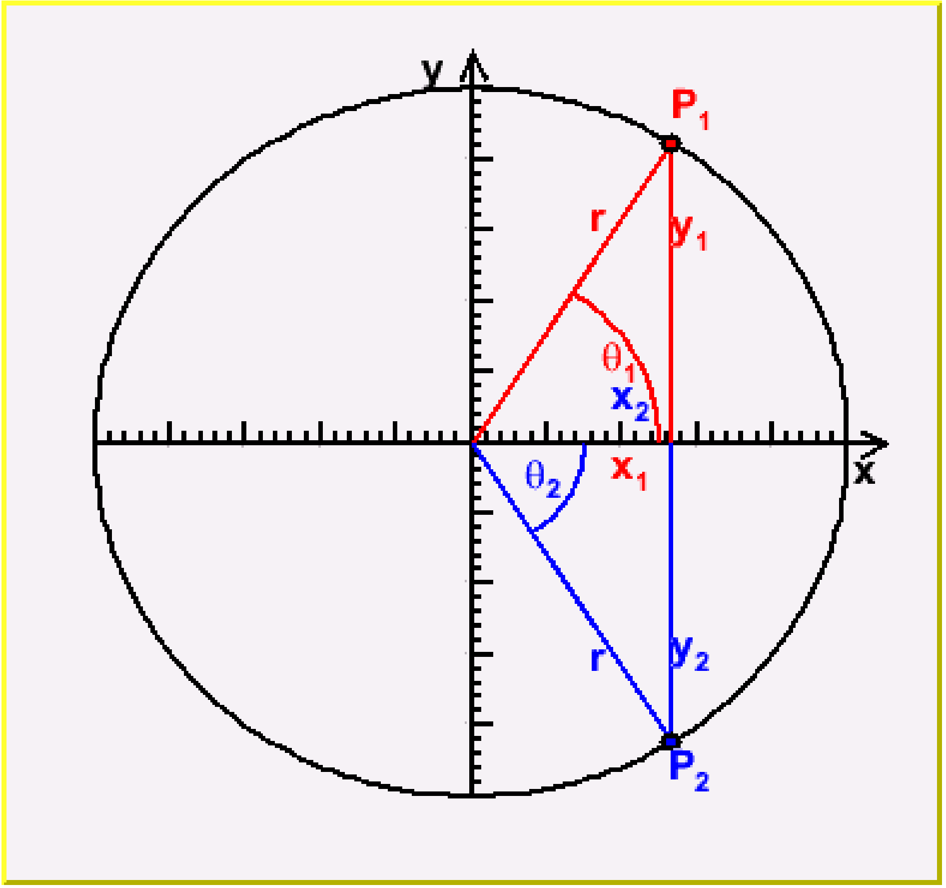
Moving from Quadrant to Quadrant: Supplementary angles (reflection about y-axis)

- $\theta_2 = 180° - \theta_1$
- $x_2 = -x_1$, $y_2 = +y_1$
- $\cos(\theta_2) = x_2 / r$
- $\sin(\theta_2) = y_2 / r$
- $\cos(180° - \theta_1) = -\cos(\theta_1)$
- $\sin(180° - \theta_1) = \sin(\theta_1)$.



Inverting the sign of an angle (reflection about x-axis)

- $\theta_2 = -\theta_1$
- $x_2 = x_1$, $y_2 = -y_1$
- $\cos(\theta_2) = x_2/r$
- $\sin(\theta_2) = y_2/r$
- $\cos(-\theta_1) = \cos(\theta_1)$
  - Cosine is an EVEN function
- $\sin(-\theta_1) = -\sin(\theta_1)$.
  - Sine is an ODD function



Complementary Angles

- $\theta_2 = 90° - \theta_1$
- $x_2 = +y_1$, $y_2 = +x_1$
- $\cos(\theta_2) = x_2/r = y_1/r$
- $\sin(\theta_2) = y_2/r = x_2/r$
- $\cos(90° - \theta_1) = \sin(\theta_1)$
- $\sin(90° - \theta_1) = \cos(\theta_1)$.
- Valid for any value of θ_1.